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Frustration without Competing Interactions

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The concept of frustration is investigated following an idea of Anderson. A simple, frustrated in Anderson's sense, nonrandom classical lattice spin system without competing interactions is discussed, which exhibits infinitely many equilibrium states at low temperature. The overlap distribution function is calculated exactly to be a delta function at zero.

KEY WORDS: Ferromagnetic spin system; infinitely many phases; frustration; overlap distribution function.

Systems with rich ground-state structure such as spin glasses have attracted much attention recently.(1) It is believed that frustration(2,3) and randomness of interactions are two necessary ingredients for producing the complicated behavior of such systems. We will discuss frustration in simple, deterministic (i.e., without random interactions), translation-invariant classical lattice spin models; for a review of periodically frustrated systems see refs. 1 and 4. The approach to frustration which appears almost exclusively in the literature is that involving competing interactions and was introduced by Toulouse.(2) It is best illustrated by the example of an antiferromagnetic nearest-neighbor-interaction spin-1/2 model on the triangular lattice. The formal Hamiltonian can be written as follows:

$$H = \sum_{i,j} \sigma_i \sigma_j \tag{1}$$

where σ_i, $\sigma_j = \pm 1$ and i and j are nearest neighbor sites on the triangular lattice. When one looks at an elementary triangle it is easy to see that at

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least one pair of spins does not minimize its interaction. Two spins align themselves in opposite directions and then the third one can minimize only one of the two remaining interactions. This choice is a source of frustration. Although it is very picturesque, this approach depends strongly on the particular choice of interactions describing the system. For example, in the above model instead of the Hamiltonian H we could use

$$H' = \sum_{\Delta} \phi_{\Delta} \tag{2}$$

where the sum is over all elementary triangles and $\phi_{\Delta} = 1/2(\sigma_i\sigma_j + \sigma_j\sigma_k + \sigma_k\sigma_i)$, where i, j, and k are the vertices of an elementary triangle Δ. Both H and H' describe the same physical system: they have the same equilibrium states. There are, however, spin configurations, the ground states, such that every three-spin interaction in H' attains its minimum. Three spins on every elementary triangle still face choices but they act collectively and therefore are not frustrated. Hamiltonians for which there exist configurations minimizing all interactions simultaneously are called minimal potentials and were introduced by Holsztynski and Slawny.[5,6] There is no known example of a translation-invariant Hamiltonian for which one cannot find an equivalent minimal potential.

There is another approach to frustration, due to Anderson.[3] He suggests the following construction. Divide the lattice into cubes of side length l. Find the configurations of two adjacent cubes with the lowest energy (with the interaction between cubes cut off). Now calculate the variance $\bar{E}^2$ of the interaction between the cubes with respect to all possible choices of the above lowest energy configurations. The spin system is then called frustrated if $\bar{E}^2$ is not proportional to the square of the area of the face of the cube:

$$\bar{E}^2/l^4 \xrightarrow[l \to \infty]{} 0 \tag{3}$$

It is interesting to note that the most popular model with competing interactions, the ANNNI model,[7] is not frustrated according to the above definition. The reason is that its ground states have one-dimensional structure. The nearest neighbor interaction along one (say x) direction is ferromagnetic and the next nearest neighbor interaction in the same direction is antiferromagnetic, which produces infinitely many ground states. The nearest neighbor ferromagnetic interactions along the y and z directions make one-dimensional ground states repeat in these directions. Any contribution to $\bar{E}^2$ across the face perpendicular to the x axis is automatically multiplied by l^4.

Now we will discuss an example of a system frustrated in Anderson's sense but without competing interactions. It is a ferromagnetic translation-invariant spin-1/2 model (Slawny model[8]) on the cubic lattice with the following Hamiltonian:

$$H = -\sum \sigma_i \sigma_j \sigma_k \sigma_m \tag{4}$$

where i, j, k, and m are vertices of a plaquette (an elementary square) and the sum is over all plaquettes. The following are ground-state configurations of this model: $(-)$ spin on the xy plane and $(+)$ spin everywhere else and similarly with the xz and yz planes. It is easy to see that every ground-state configuration can be generated using translates of these three configurations and superpositions [two superposed $(-)$ spins create $(+)$ spin]. There is an infinite (continuum) number of them and in each, every plaquette contains an even number of $(-)$ spins having the smallest possible energy; the interaction is a minimal potential. Nevertheless, the system is frustrated. Using the fact that any ferromagnetic Hamiltonian is invariant under a spin flip associated with any of its ground-state configurations, to calculate $\bar{E}^2$ one can fix all $(+)$ spins in one box and vary the ground-state configurations in the other one. Let us compute $\bar{E}^2$ across the face perpendicular to the x axis. A part of a ground-state configuration in one box interacting with the $(+)$ configuration in the other one may be seen as generated by planes with $(-)$ spins and perpendicular to the y and z axes. Let k and d denote the number of pairs of the nearest neighbor parallel planes such that all spins are flipped in one of them only. This raises the energy of the interaction across the face of a cube of side length l by $2(l+1)(k+d)$. Taking into account all $2^l 2^l$ choices of different planes, we obtain

$$\bar{E}^2 = 1/2^{2l} \sum_{k=0}^{l} \sum_{d=0}^{l} \binom{l}{k}\binom{l}{d} [2(l+1)\, l - 2(l+1)(k+d)]^2 \tag{5}$$

and after some algebra

$$\bar{E}^2 = 2(l+1)^2\, l \tag{6}$$

We have shown that a very simple deterministic model without competing interactions is frustrated in Anderson's sense. Below we will show that it has a very rich structure of low-temperature equilibrium states while having a trivial overlap distribution function.

Using the general theory of ferromagnetic systems developed by Holsztynski and Slawny,[6,9–12] one can show that every ground-state configuration corresponds to a low-temperature equilibrium state which is a small perturbation. Therefore, we have many complicated low-temperature magnetic structures, not only the simple periodic ones observed in all periodically frustrated spin systems studied so far.[1] Let ρ_0 be the infinite-volume limit of the Gibbs state with free boundary conditions. In its decomposition into extremal states every Gibbs state corresponding to one of the above-described ground-state configurations contributes with the same weight. Having complete information about low-temperature equilibrium states and the decomposition of ρ_0 one can compute explicitly the overlap distribution function $P(q)$, an order parameter introduced by Parisi[13]:

$$P(q) = \sum_{\alpha\beta} p_\alpha p_\beta \delta(q - q_{\alpha\beta}) \tag{7}$$

where

$$q_{\alpha\beta} = \lim_{l \to \infty} 1/l^3 \sum_i \langle \sigma_i \rangle_\alpha \langle \sigma_i \rangle_\beta \tag{8}$$

and p_α is the weight of the pure phase $\langle \cdot \rangle_\alpha$ in the decomposition of ρ_0 into extremal Gibbs states. Because of the above-mentioned symmetries of our ferromagnetic Hamiltonian, to compute $P(q)$ it is enough to consider overlaps between the configuration with all $(+)$ spins and all other ground-state configurations. If a ground-state configuration is generated by superposition of d, k, and s planes with $(-)$ spins, and perpendicular respectively to x, y, and z axes, the overlap is equal to $(l-2d)(l-2k)(l-2s)/l^3$. Now we will prove that $P(q) = \delta(q)$, the delta distribution at 0. It is enough to show that $\int_r^t P(q)\, dq = 0$ for every $0 < r < t \leqslant 1$. We have

$$\int_r^t P(q)\, dq \leqslant \lim_{l \to \infty} 6/2^{3l} l^3 \sum_{d=0}^{(1-r)l/2} \sum_{k=0}^{l} \sum_{s=0}^{l} \binom{l}{d}\binom{l}{k}\binom{l}{s} (l-2d)(l-2k)(l-2s) = 0 \tag{9}$$

The last equality follows from the fact that the binomial random variable is asymptotically normal. In a more general context the triviality of the overlap distribution function follows from the ergodicity of the Haar measure present in (7).[14] The overlap distribution in our example is a delta distribution even in the presence of many phases. It was pointed out recently by Huse and Fisher[15] that the overlap distribution is not a very reliable indicator of the number of pure phases in the system.

In conclusion, we note that the nonrandom model frustrated in

Anderson's sense but without competing interactions and with an infinite number of low-temperature phases was investigated. In the presence of an external magnetic field all equilibrium states of this model except the ferromagnetic one disappear. It would be interesting to construct a non-random model with many phases even in the presence of an external magnetic field and then to compare it with the behavior of a field-cooled spin glass.

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REFERENCES

1. K. Binder and A. P. Young, *Rev. Mod. Phys.* **58**:801 (1986).
2. G. Toulouse, *Commun. Phys.* **2**:115 (1977).
3. P. W. Anderson, *J. Less-Common Metals* **62**:291 (1978).
4. R. Liebmann, *Statistical Mechanics of Periodic Frustrated Ising Systems* (Springer, Berlin, 1986).
5. W. Holsztynski and J. Slawny, *Commun. Math. Phys.* **61**:177 (1978).
6. J. Slawny, Low temperature properties of classical lattice systems: Phase transitions and phase diagrams, in *Phase Transitions and Critical Phenomena*, Vol. 11, C. Domb and J. L. Lebowitz, eds. (Academic Press, New York, 1987).
7. M. E. Fisher and W. Selke, *Phil. Trans. R. Soc.* **302**:1 (1981).
8. J. Slawny, *Commun. Math. Phys.* **35**:297 (1974).
9. W. Holsztynski and J. Slawny, *Commun. Math. Phys.* **66**:147 (1979).
10. J. Slawny, Ferromagnetic systems with local symmetries, unpublished (1979).
11. J. Miękisz, *Commun. Math. Phys.* **109**:353 (1987).
12. J. Miękisz, *J. Phys. A: Math. Gen.* **21**:1679 (1988).
13. G. Parisi, *Phys. Rev. Lett.* **43**:1946 (1983).
14. J. Slawny, Private communication (1988).
15. D. A. Huse and D. S. Fisher, *J. Phys. A: Math. Gen.* **20**:L997 (1987).